1.

In this case the period $T_0 = 2$. Hence

$$\omega_0 = \frac{2\pi}{2} = \pi$$

and

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi t + b_n \sin n\pi t$$

where

$$f(t) = \begin{cases} 2At & |t| \le \frac{1}{2} \\ 2A(1-t) & \frac{1}{2} < t \le \frac{3}{2} \end{cases}$$

Here it will be advantageous to choose the interval of integration from $-\frac{1}{2}$ to $\frac{3}{2}$ rather than 0 to 2.

A glance at Fig. 3.9a shows that the average value (dc) of $f(t)$ is zero, so that $a_0 = 0$. Also

$$\begin{aligned} a_n &= \frac{2}{2} \int_{-1/2}^{3/2} f(t) \cos n\pi t \, dt \\ &= \int_{-1/2}^{1/2} 2At \cos n\pi t \, dt + \int_{1/2}^{3/2} 2A(1-t) \cos n\pi t \, dt \end{aligned}$$

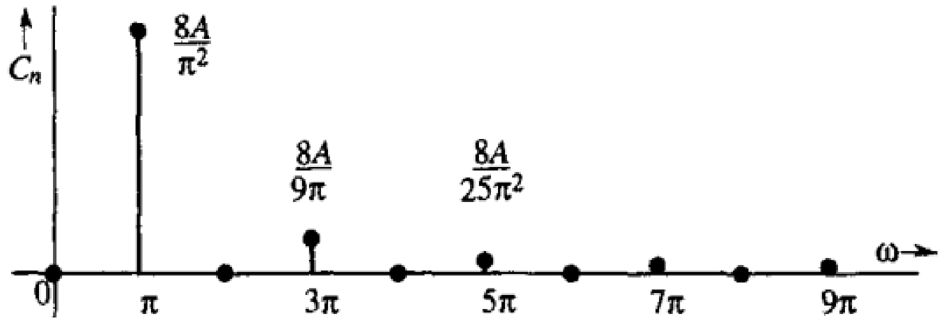
The detailed evaluation of the above integrals shows that both have a value of zero. Therefore

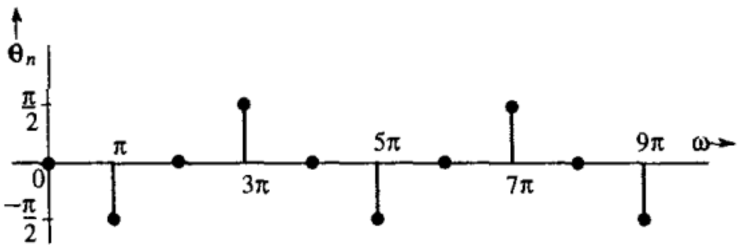
$$a_n = 0 \tag{3.62a}$$

$$b_n = \int_{-1/2}^{1/2} 2At \sin n\pi t \, dt + \int_{1/2}^{3/2} 2A(1-t) \sin n\pi t \, dt$$

The detailed evaluation of these integrals yields

$$b_n = \frac{8A}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n \text{ even} \\ \frac{8A}{n^2\pi^2} & n = 1, 5, 9, 13, \cdots \\ -\frac{8A}{n^2\pi^2} & n = 3, 7, 11, 15, \cdots \end{cases} \tag{3.62b}$$





(c)

Fig. 3.9 A triangular periodic signal and its Fourier spectra.

Therefore

$$f(t) = \frac{8A}{\pi^2}\left[\sin\,\pi t - \frac{1}{9}\sin\,3\pi t + \frac{1}{25}\sin\,5\pi t - \frac{1}{49}\sin\,7\pi t + \cdots\right] \tag{3.63}$$

In order to plot Fourier spectra, the series must be converted into compact trigonometric form as in Eq. (3.54). In this case, sine terms are readily converted into cosine terms with a suitable phase shift. For example,

$$\pm\sin\,kt = \cos\,(kt \mp 90^\circ)$$

Using this identity, Eq. (3.63) can be expressed as

$$f(t) = \frac{8A}{\pi^2}\left[\cos\,(\pi t - 90^\circ) + \frac{1}{9}\cos\,(3\pi t + 90^\circ) + \frac{1}{25}\cos\,(5\pi t - 90^\circ) + \frac{1}{49}\cos\,(7\pi t + 90^\circ) + \cdots\right] \tag{3.64}$$

In this series all the even harmonics are missing. The phases of odd harmonics alternate from -90° to 90°. Figure 3.9 shows amplitude and phase spectra for $f(t)$. ■

2.

The Fourier series is

$$w(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos\,n\omega_0 t + b_n\sin\,n\omega_0 t$$

where

$$a_0 = \frac{1}{T_0}\int_{T_0} w(t)\,dt$$

In the preceding equation, we may integrate $w(t)$ over any interval of duration T_0. Figure 2.22a shows that the best choice for a region of integration is from $-T_0/2$ to $T_0/2$. Because $w(t) = 1$ only over $(-T_0/4,\ T_0/4)$ and $w(t) = 0$ over the remaining segment,

$$a_0 = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} dt = \frac{1}{2} \tag{2.74a}$$

We could have found a_0, the average value of $w(t)$, to be $1/2$ merely by inspection of $w(t)$ in Fig. 2.22a. Also,

$$a_n = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} \cos n\omega_0 t \, dt = \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ \dfrac{2}{\pi n} & n = 1, 5, 9, 13, \cdots \\ -\dfrac{2}{\pi n} & n = 3, 7, 11, 15, \cdots \end{cases} \tag{2.74b}$$

$$b_n = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} \sin nt \, dt = 0 \tag{2.74c}$$

In these derivations we used the fact that $\omega_0 T_0 = 2\pi$. Therefore,

$$w(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos \omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \cdots\right) \tag{2.75}$$

Observe that $b_n = 0$ and all the sine terms are zero. Only the cosine terms appear in the trigonometric series. The series is therefore already in compact form, except that the amplitudes of alternating harmonics are negative. Now by definition, amplitudes C_n are positive [see Eq. (2.67a)]. The negative sign can be accommodated by a phase of π radians. This can be seen from the trigonometric identity*

$$-\cos x = \cos(x - \pi)$$

Using this fact, we can express the series in Eq. (2.75) as

$$w(t) = \frac{1}{2} + \frac{2}{\pi}\left[\cos \omega_0 t + \frac{1}{3}\cos(3\omega_0 t - \pi) + \frac{1}{5}\cos 5\omega_0 t + \frac{1}{7}\cos(7\omega_0 t - \pi) + \frac{1}{9}\cos 9\omega_0 t + \cdots\right]$$

This is the desired form of the compact trigonometric Fourier series. The amplitudes are

$$C_0 = \frac{1}{2}$$

$$C_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{2}{\pi n} & n \text{ odd} \end{cases}$$

$$\theta_n = \begin{cases} 0 & \text{for all } n \neq 3, 7, 11, 15, \cdots \\ -\pi & n = 3, 7, 11, 15, \cdots \end{cases}$$

We could plot amplitude and phase spectra using these values. We can, however, simplify our task in this special case if we allow amplitude C_n to take on negative values. If this is allowed, we do not need a phase of $-\pi$ to account for the sign. This means the phases of all components are zero, and we can discard the phase spectrum and manage with only the amplitude spectrum, as shown in Fig. 2.22b. Observe that there is no loss of information in doing so and that the amplitude spectrum in Fig. 2.22b has the complete information about the Fourier series in Eq. (2.75). Therefore, *whenever all sine terms vanish ($b_n = 0$), it is convenient to allow C_n to take on negative values*. This permits the spectral information to be conveyed by a single spectrum—the amplitude spectrum. Because C_n can be positive as well as negative, the spectrum is called the **amplitude spectrum** rather than the magnitude spectrum.

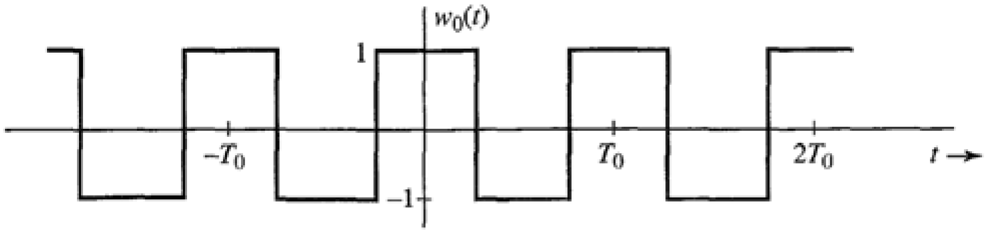


Figure 2.23 Bipolar square pulse periodic signal.

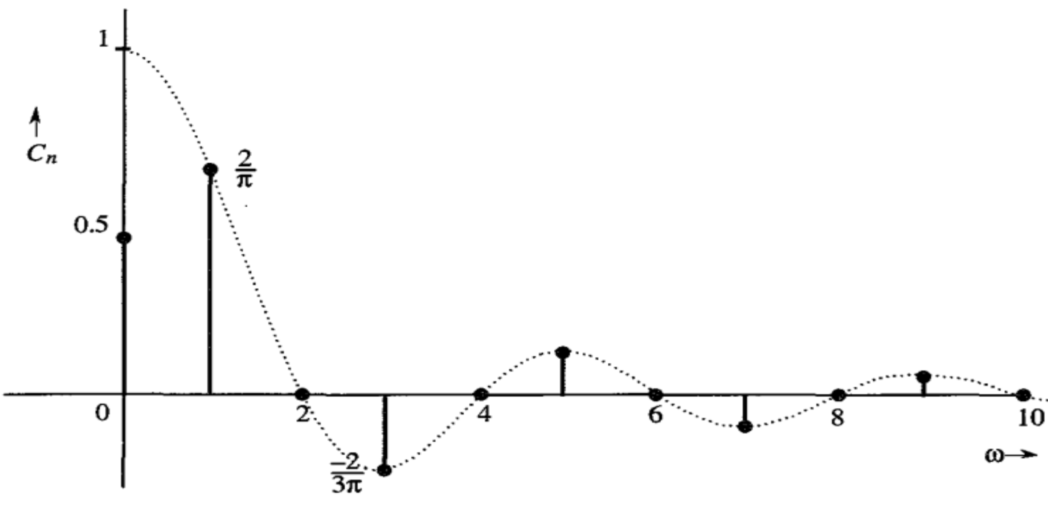
Another useful function that is related to the periodic square wave is the bipolar square wave $w_0(t)$ shown in Fig. 2.23a. We encounter this signal in switching applications. Note that $w_0(t)$ is basically $w(t)$ minus its dc component. It is easy to see that

$$w_0(t) = 2[w(t) - 0.5]$$

Hence, from Eq. (2.75) it follows that

$$w_0(t) = \frac{4}{\pi}\left(\cos\,\omega_0 t - \frac{1}{3}\cos\,3\omega_0 t + \frac{1}{5}\cos\,5\omega_0 t - \frac{1}{7}\cos\,7\omega_0 t + \cdots\right) \quad (2.76)$$

Comparison of this equation with Eq. (2.75) shows that the Fourier components of $w_0(t)$ are identical to those of $w(t)$ [Eq. (2.75)] in every respect except for doubling the amplitudes and loss of dc.

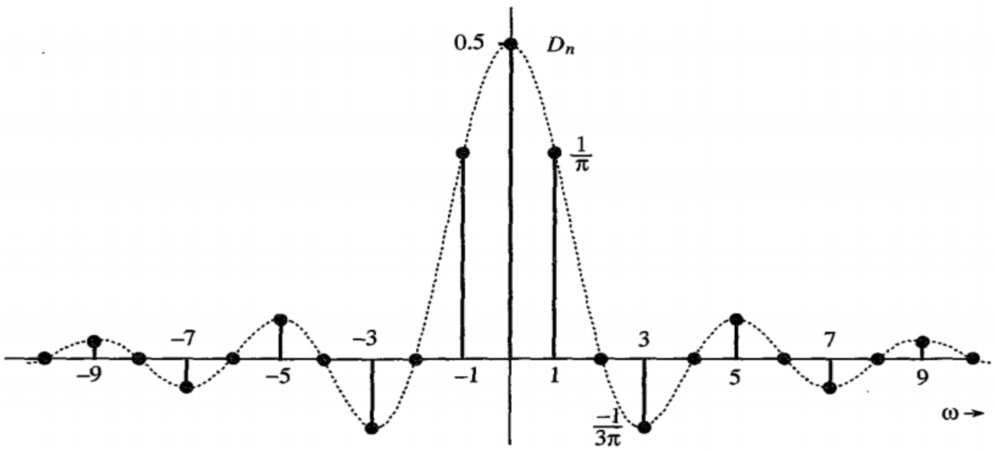


We have

$$w(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

where

$$\begin{aligned} D_n &= \frac{1}{T_0}\int_{T_0} w(t) e^{-jn\omega_0 t}\, dt \\ &= \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} e^{-jn\omega_0 t}\, dt \\ &= \frac{1}{-jn\omega_0 T_0}\left(e^{-jn\omega_0 T_0/4} - e^{jn\omega_0 T_0/4}\right) \\ &= \frac{2}{n\omega_0 T_0}\sin\left(\frac{n\omega_0 T_0}{4}\right) = \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) \end{aligned}$$



3.

The trigonometric Fourier series for $\delta_{T_0}(t)$ is given by

$$\delta_{T_0}(t) = C_0 + \sum C_n \cos(n\omega_0 t + \theta_n) \qquad \omega_0 = \frac{2\pi}{T_0}$$

We first compute a_0, a_n, and b_n:

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\, dt = \frac{1}{T_0}$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t) \cos n\omega_0 t\, dt = \frac{2}{T_0}$$

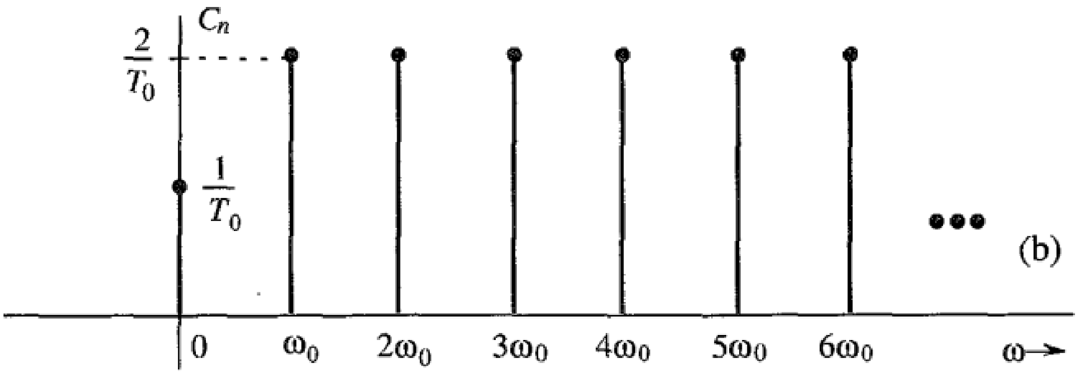
This result follows from the sampling property (2.19) of the impulse function. Similarly, using the sampling property of the impulse, we obtain

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t) \sin n\omega_0 t\, dt = 0$$

Therefore, $C_0 = 1/T_0$, $C_n = 2/T_0$, and $\theta_n = 0$. Thus,

$$\delta_{T_0}(t) = \frac{1}{T_0}\left(1 + 2\sum_{n=1}^{\infty} \cos n\omega_0 t\right) \tag{2.77}$$

Figure 2.24b shows the amplitude spectrum. The phase spectrum is zero.



The exponential Fourier series is given by

$$\delta_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

where

$$D_n = \frac{1}{T_0} \int_{T_0} \delta_{T_0}(t) e^{-jn\omega_0 t}\, dt$$

Choosing the interval of integration $(-T_0/2,\ T_0/2)$ and recognizing that over this interval $\delta_{T_0}(t) = \delta(t)$,

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jn\omega_0 t}\, dt$$

In this integral, the impulse is located at $t = 0$. From the sampling property of the impulse function, the integral on the right-hand side is the value of $e^{-jn\omega_0 t}$ at $t = 0$ (where the impulse is located). Therefore,

$$D_n = \frac{1}{T_0} \tag{2.88}$$

and

$$\delta_{T_0}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0} \tag{2.89}$$

Equation (2.89) shows that the exponential spectrum is uniform ($D_n = 1/T_0$) for all the frequencies, as shown in Fig. 2.27. The spectrum, being real, requires only the amplitude plot. All phases are zero. Compare this spectrum to the trigonometric spectrum shown in Fig. 2.24b. The dc components are identical and the exponential spectrum amplitudes are half those in the trigonometric spectrum for all $\omega > 0$.